

Dyon–Oscillator Duality ^{*}

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Abstract

The dyon–oscillator duality presented in this lecture can be treated as a prototype of the Seiberg–Witten duality in nonrelativistic quantum mechanics. The key statement declares that in some spatial dimensions the oscillator-like systems are dual to the atoms composed of the electrical charged particle and dyon, i.e., monopoles provided by both magnetic and electric charge.

1 Introduction

The objective of the present lecture is to illustrate the property of the Schrödinger equation which is called here the dyon–oscillator duality. The property is in the following. The Schrödinger equation for an oscillator possesses two parameters – the energy E and the cyclic frequency ω . The quantization leads to the constraint $E = \hbar\omega(N + D/2)$ where $N = 0, 1, 2, \dots$, and D is the dimension of the configuration space of the oscillator. If ω is fixed, then E is quantized and that is the standard situation. Imagine for a moment that now E is fixed. Whence, necessarily ω is quantized, and we are in a nonstandard situation. The question is whether the nonstandard situation corresponds to any physics, i.e., whether it is possible to find such a transformation that converts the oscillator into a physical system with a coupling constant α , being a function of E , and energy ε , depending on ω . If there exists such a transformation, we can confirm that the "nonstandard oscillator" is identical to that physical system. Below will be shown the validity of the described picture for dimensions $D = 1, 2, 4, 8$ and that the final system is a bound system of charge–dyon (remind, that dyon is the hypothetical particle introduced by Schwinger which is unlike the Dirac monopole, endowed with not just magnetic but electric charge as well). As the "standard" and "nonstandard" regimes are mutually exclusive, the initial oscillator and

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the final "charge-dyon" system are dual to each other, and that explains the relevancy of the term "dyon-oscillator duality". Note also that in the initial system the spectrum is discrete only, i.e. the particle has just a finite motion (for such cases it is usually said that we have a model with confinement). Generally speaking, the spectrum of the final system includes the discrete spectrum as well as the continuous one, i.e. in that model there is no confinement. However, unlike the first model, in the second model we have monopoles. There is some analogy between the dyon-oscillator and the Seiberg-Witten duality, according to which the gauge theories with strong interactions are equivalent to the theories having weak interaction on the one hand and topological nontrivial objects, such as monopoles and dyons are, on the other hand.

2 Radial Equations

Let consider the equation

$$\frac{d^2 R}{du^2} + \frac{D-1}{u} \frac{dR}{du} - \frac{L(L+D-2)}{u^2} R + \frac{2\mu}{\hbar^2} \left(E - \frac{\mu\omega^2 u^2}{2} \right) R = 0. \quad (1)$$

Here R is the radial part of the wave function for the D -dimensional oscillator ($D > 2$) and $L = 0, 1, 2, \dots$ are the eigenvalues of the global angular momentum.

Introduce $r = u^2$ and take into account that

$$\frac{1}{u} \frac{d}{du} = 2 \frac{d}{dr}, \quad \frac{d^2}{du^2} = 2 \frac{d}{dr} + 4r \frac{d^2}{dr^2}.$$

Then, equation (1) transforms into

$$\frac{d^2 R}{dr^2} + \frac{d-1}{r} \frac{dR}{dr} - \frac{l(l+d-2)}{r^2} R + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{\alpha}{r} \right) R = 0, \quad (2)$$

where

$$d = D/2 + 1, \quad l = L/2, \quad (3)$$

$$\varepsilon = -\mu\omega^2/8, \quad \alpha = E/4. \quad (4)$$

This is quite an unexpected result. If $D = 4, 6, 8, 10, \dots$, then $d = 3, 4, 5, 6, \dots$, and equation (2) is formally identical to the radial equation for a d -dimensional hydrogen atom (for odd $D > 2$ the value of d is half-integer and so cannot have the meaning of the dimension of the space in a usual sense). Then, l takes not just integer but half-integer values as well, and a question arises about the origin of the fermion degree of freedom. The answer to the question will be given later. Finally, as has been mentioned in the first section, equations (1) and (2) are dual to each other and the duality transformation is $r = u^2$.

Earlier, just the radial part of the wave function of the oscillator was considered. For the Schrödinger equation we must take into account the angular part as well. Thus, the duality transformation must also include the transformation of angular variables. If we interpret the change of variables $r = u^2$ as a mechanism of generation of electric charge,

then (as will be shown later) the transformation of some angular variables is responsible for the generation of magnetic charges.

In the next sections, we study dimensions $D = 1$ and $D = 2$ not considered in equation (1). Then, we analyze the dimensions $D = 4$ and $D = 8$. The dyon–oscillator duality is limited to these four dimensions. We postpone for a while the discussion of the problem of selection of the dimensions $D = 1, 2, 4, 8$.

3 1D Coulomb Anyon

Consider the one-dimensional Schrödinger equation

$$\frac{d^2\Psi}{du^2} + \frac{2\mu}{\hbar^2} \left(E - \frac{\mu\omega^2 u^2}{2} \right) \Psi = 0, \quad (5)$$

where $-\infty < u < \infty$. We define a new variable

$$x = u^2,$$

and using the identity

$$\frac{d^2}{du^2} = 4|x| \left(\frac{d^2}{dx^2} + \frac{1}{2x} \frac{d}{dx} \right)$$

and setting

$$\Psi = C x^{-1/4} \Phi, \quad (6)$$

arrive at the equation

$$\frac{d^2\Phi}{dx^2} + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{\alpha}{|x|} + \frac{\hbar^2}{2\mu} \frac{3}{16x^2} \right) \Phi = 0, \quad (7)$$

where ε and α are the same as in (4).

Let us introduce the quantity ν which takes two values: $\nu = 1/4$ and $\nu = 3/4$, and rewrite the last equation in the form

$$\frac{d^2\Phi^{(\nu)}}{dx^2} + \frac{2\mu}{\hbar^2} (\varepsilon - V_c - V_{cs}) \Phi^{(\nu)} = 0, \quad (8)$$

where $V_c = -\alpha/|x|$ and V_{cs} is the Calogero–Sutherland potential $V_{cs} = -\hbar^2\nu(1-\nu)/2\mu x^2$.

In one spatial dimension, a particle moving in the Calogero–Sutherland potential has a very unusual property. Unlike the potential V_{cs} , the wave function is not invariant under the replacement $\nu \rightarrow (1-\nu)$. It describes a boson for even ν and a fermion for odd ν . Statistics corresponding to the other values of ν is called the fractional statistics, and the system influenced along with V_{cs} by a potential binding the particle to the center is called the 1D anyon. So, we have started from the 1D quantum oscillator and arrived at the 1D Coulomb anyon.

Comparing Eq. (5) with Eq. (8), we summarize that there are two alternative possibilities connected with Eq. (5) – explicit and hidden. In the first case, the parameter

ω is fixed ($\omega = \text{fix.} > 0$) and plays a role of the coupling constant, the parameter E is quantized and has the meaning of energy, and the system is the 1D quantum oscillator. For a hidden possibility, the parameter E is fixed ($E = \text{fix.} > 0$), the coupling constant is equal to $E/4$, ω is quantized, the quantity $\varepsilon = -\mu\omega^2/8$ takes the meaning of energy, and the system is the 1D Coulomb anyon. Since the 1D Coulomb anyon includes the $1/x^2$ interaction, it pretends to be a magnetic monopole in one spatial dimension. So, the anyon-oscillator duality is a prototype of the dyon-oscillator duality in 1D Quantum Mechanics.

Now we can calculate the energy levels ε_n and the wave functions $\Phi_n^{(\nu)}$ in the following way. For energy levels we have

$$\varepsilon = -\frac{\mu\omega^2}{8} = -\frac{\mu}{8} \left[\frac{E}{\hbar(2n+2\nu)} \right]^2 = -\frac{\mu}{8} \left[\frac{4\alpha}{\hbar(2n+2\nu)} \right]^2 = -\frac{\mu\alpha^2}{2\hbar^2(n+\nu)^2},$$

where $N = 2n + 2\nu - 1/2$ with N numerating the energy levels $E = \hbar\omega(N + 1/2)$ and n being integer and nonnegative.

Consider the wave functions. It follows from (6) that

$$\Phi_n^{(\nu)} = \frac{1}{C} x^{1/4} \Psi_n^{(\nu)},$$

where $\Psi_n^{(\nu)} \equiv \Psi$, and therefore

$$\int_{-\infty}^{\infty} |\Phi_n^{(\nu)}|^2 dx = \frac{1}{|C|^2} \int_{-\infty}^{\infty} x^{1/2} |\Psi_n^{(\nu)}|^2 dx.$$

The integral in the left-hand side is equal to 1, from which it follows that

$$|C|^2 = \int_{-\infty}^{\infty} u^2 |\Psi_N(u)|^2 du = \overline{u^2} = \frac{2(n+\nu)\hbar}{\mu\omega}.$$

Thus,

$$\Phi_n^{(\nu)} = \frac{(-1)^n}{\sqrt{2}} \sqrt{\frac{\mu\omega}{\hbar(n+\nu)}} x^{1/4} \Psi_n^{(\nu)}$$

if we choose the phase factor as $(-1)^n$.

Remind that according to the theory of quantum oscillator,

$$\Psi_N^{(\nu)} = \left(\frac{\mu\omega}{\pi\hbar} \right)^{1/4} \frac{1}{2^N N!} e^{-\mu\omega u^2/2} H_N \left(u \sqrt{\frac{\mu\omega}{\hbar}} \right),$$

where $H_N(\xi)$ is the Hermite polynomial

$$H_N(\xi) = (-1)^N e^{\xi^2} \frac{d^N}{d\xi^N} e^{-\xi^2}.$$

Further, it is known that Hermite polynomials could be expressed in terms of confluent hypergeometric functions. For our case ($s = 0, 1/2$)

$$H_{2n+2s}(z) = (-1)^n \frac{(2n+2s)!}{n!} (2z)^{2s} F(-n, 2s+1/2, z^2).$$

Using the identification $y = x\mu\omega/\hbar$ and the relations $2s+1/2 = 2\nu$ and $\mu\omega/\hbar = 2\mu\alpha/\hbar^2(n+\nu)$, we get

$$\Phi_n^{(\nu)} = \sqrt{\frac{\mu\alpha}{\hbar^2}} \frac{1}{2^{n-\nu+1/4}} \frac{\sqrt{\Gamma(2n+2\nu+1/2)}}{\pi^{1/4} n! (n+\nu)} y^\nu e^{-|y|/2} F(-n, 2\nu, y),$$

and after taking into account the duplication formula for Euler's gamma-function

$$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z+1/2)$$

we arrive at the formula

$$\Phi_n^{(\nu)} = \frac{\sqrt{\mu\alpha}}{\hbar} \frac{1}{n+\nu} \frac{1}{\Gamma(2\nu)} \sqrt{\frac{\Gamma(n+2\nu)}{n!}} y^\nu e^{-|y|/2} F(-n, 2\nu, y).$$

So, we have two types of 1D Coulomb anyons with $\nu = 1/4$ and $\nu = 3/4$, respectively.

4 Magnetic Vortex

Now turn to the cyclic oscillator. Here is the first example where along with the radial variable there appears an angular one. In the polar coordinates (u, φ) , where $0 \leq u < \infty$, $0 \leq \varphi < 2\pi$, the Schrödinger equation takes the form

$$\frac{\partial^2 \Psi}{\partial u^2} + \frac{1}{u} \frac{\partial \Psi}{\partial u} + \frac{1}{u^2} \frac{\partial^2 \Psi}{\partial \varphi^2} + \frac{2\mu}{\hbar^2} \left(E - \frac{\mu\omega^2 u^2}{2} \right) \Psi = 0. \quad (9)$$

Input new variables

$$r = u^2, \quad \phi = 2\varphi \quad (10)$$

and rewrite equation (9) as

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{\alpha}{r} \right) \Psi = 0. \quad (11)$$

where ε and α are given by expressions (4). Equation (10) is identical to the Schrödinger equation for a two-dimensional hydrogen atom; however, $\phi \in [0, 4\pi)$. Thus, instead of a plane, we have two-sheeted Riemann surface. As a consequence, the single-valuedness condition $\Psi(r, \phi + 4\pi) = \Psi(r, \phi)$ leads to the integer as well as half-integer eigenvalues for the angular momentum. The solution of the first type for $\phi \rightarrow (\phi + 2\pi)$ does not change the sign, while the second type solutions under the same transformation change the sign. Whence, without loss of information we can think of $\Psi(r, \phi)$ defined in the region $0 \leq$

$\phi < 2\pi$ and having two modifications that differ from each other by the quantum number $s = 0$ or $1/2$. In addition $\Psi^{(0)}(r, \phi + 2\pi) = \Psi^{(0)}(r, \phi)$ and $\Psi^{(1/2)}(r, \phi + 2\pi) = -\Psi^{(1/2)}(r, \phi)$. We say that these wave functions describe the system with full inner momentum $s = 0$ and $s = 1/2$, respectively.

Introduce now the important substitution

$$\Psi^{(s)}(r, \phi) = e^{is\phi} \bar{\Psi}^{(s)}(r, \phi), \quad (12)$$

where $\bar{\Psi}^{(s)}(r, \phi + 2\pi) = \bar{\Psi}^{(s)}(r, \phi)$ for $s = 0$ as well as for $s = 1/2$. From (11) and (12) it follows that the function $\bar{\Psi}^{(s)}$ satisfies the equation

$$\frac{\partial^2 \bar{\Psi}^{(s)}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\Psi}^{(s)}}{\partial r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \phi} + is \right)^2 \bar{\Psi}^{(s)} + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{\alpha}{r} \right) \bar{\Psi}^{(s)} = 0. \quad (13)$$

Now let us clear up to what system there corresponds equation (13). Input the Cartesian coordinates

$$x_1 = r \cos \phi, \quad x_2 = r \sin \phi.$$

As $\partial/\partial \phi = x_1 \partial/\partial x_2 - x_2 \partial/\partial x_1$, then instead of (13) we have

$$\left(\frac{\partial}{\partial x_1} - \frac{isx_2}{r^2} \right)^2 \bar{\Psi}^{(s)} + \left(\frac{\partial}{\partial x_2} + \frac{isx_1}{r^2} \right)^2 \bar{\Psi}^{(s)} + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{\alpha}{r} \right) \bar{\Psi}^{(s)} = 0. \quad (14)$$

To this equation there corresponds the Hamiltonian

$$\hat{H} = \frac{1}{2\mu} \left[\left(\hat{p}_1 - \frac{\hbar s x_2}{r^2} \right)^2 + \left(\hat{p}_2 + \frac{\hbar s x_1}{r^2} \right)^2 \right] - \frac{\alpha}{r}. \quad (15)$$

Input a vector

$$\vec{A} = \frac{g}{r^2} (x_2, -x_1)$$

where $g = \hbar cs/e$ and $e = \sqrt{\alpha}$. As

$$\text{rot} \vec{A} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = g \left[\frac{\partial}{\partial x_1} \left(\frac{\partial}{\partial x_1} \frac{1}{r} \right) + \frac{\partial}{\partial x_2} \left(\frac{\partial}{\partial x_2} \frac{1}{r} \right) \right] = g \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \frac{1}{r} = 2\pi g \delta(\vec{x}),$$

then \vec{A} is the vector potential created by the magnetic vortex of the magnetic charge g and placed in the origin of coordinates.

Now instead of (15) we get the Hamiltonian

$$\hat{H} = \frac{1}{2\mu} \left(\hat{p}_\mu - \frac{e}{c} A_\mu \right)^2 - \frac{e^2}{r},$$

corresponding to the two-dimensional charge-dyon system. So it is proved that the cyclic oscillator is dual to the charge-dyon system, being a generalization of a usual two-dimensional hydrogen atom.

Let us discuss the correspondence between the cyclic oscillator and the charge–dyon system in detail. It is well-known that in the polar coordinates (u, φ) the energy and wave function of the cyclic oscillator are given by the formulas $E = \hbar\omega(2n + |M| + 1)$ and

$$\Psi_{n,M}(u, \varphi) = A_{n,M} u^{|M|} e^{-\mu\omega u^2/\hbar} F\left(-n, |M| + 1, \frac{\mu\omega}{\hbar} u^2\right) e^{iM\varphi},$$

where $n = 0, 1, 2, \dots$, $M = 0, \pm 1, \pm 2, \dots$. To even and odd wave functions there correspond even and odd values of M . Formally, the allowance for parity can be realized by introducing the quantum numbers $s = 0, 1/2$ and $m = 0, \pm 1, \pm 2, \dots$, so that $M = 2(m + s)$.

The energy ε is calculated similarly to the one in the previous section

$$\varepsilon = -\frac{\mu e^4}{2\hbar^2(n + |m + s| + 1/2)^2}. \quad (16)$$

Next, going over to $r = u^2$ and $\phi = 2\varphi$ we have

$$\Psi_{n,m}^{(s)} = A_{n,m}^{(s)} r^{|m+s|} e^{-\mu\omega r/\hbar} F(-n, 2|m + s| + 1, \mu\omega r/\hbar) e^{i(m+s)\phi}.$$

It remains to pass from the two-sheeted Riemann surface ($0 \leq \phi < 4\pi$) to the plane ($0 \leq \phi < 2\pi$), then take into account (12) and the last formula with the expression $\mu\omega/\hbar = 2\mu e^2/\hbar^2(n + |m + s| + 1/2)$ and, introducing a new variable $\rho = 2\mu e^2 r/\hbar^2(n + |m + s| + 1/2)$, write

$$\overline{\Psi}_{n,m}^{(s)}(\rho, \phi) = C_{n,m}^{(s)} \rho^{|n+m|} e^{-\rho} F(-n, 2|n + m| + 1, \rho) e^{im\phi}, \quad (17)$$

where the normalization constant $C_{n,m}^{(s)}$ is determined by the condition

$$2\pi \int_0^\infty \left| \overline{\Psi}_{n,m}^{(s)}(\rho, \phi) \right|^2 r dr = 1.$$

Go back to the transformation (10) and pass there from the polar coordinates (r, ϕ) to the Cartesian ones (x_1, x_2) . Note that $\phi \in [0, 4\pi)$. We have

$$\begin{aligned} x_1 &= r \cos \phi = u^2 [\cos^2(\phi/2) - \sin^2(\phi/2)] = u_1^2 - u_2^2, \\ x_2 &= r \sin \phi = 2u^2 \sin^2(\phi/2) \cos(\phi/2) = 2u_1 u_2. \end{aligned} \quad (18)$$

This transformation is known from celestial mechanics as the Levi–Civita transformation. In terms of the complex coordinates $z = x_1 + ix_2$, $v = u_1 + iu_2$ it takes the form $z = v^2$, i.e., corresponds to the square of the complex variable v . The Levi–Civita transformation together with the transformations (10) and the \mathbf{Z}_2 -reduction compose the duality transformation. Note that $x = \sqrt{x_1^2 + x_2^2} = u^2 \equiv u_1^2 + u_2^2$. The last expression is known as the Euler’s identity. Thus, the Levi–Civita transformation is bilinear coordinate transformation obeying Euler’s identity. This fact is quite noteworthy from the mathematical point of view, and we will have an opportunity to discuss it.

5 Charge–Dyon System

Unlike the two-dimensional space, in four-dimensions there are several types of "spherical coordinates". We take the ones used in the theory of symmetrical top

$$u_1 + iu_2 = u \cos(\beta/2) e^{i(\alpha+\gamma)/2}, \quad u_3 + iu_4 = u \sin(\beta/2) e^{i(\alpha-\gamma)/2}. \quad (19)$$

For $u = \text{const}$, the position on a sphere is parametrized by the coordinates (α, β, γ) that cover the sphere completely when

$$\alpha \in [0, 2\pi), \quad \beta \in [0, \pi), \quad \gamma \in [0, 4\pi).$$

In the coordinates (19), the length-element and the Laplacian are given by

$$dl^2 = du^2 + \frac{u^2}{4} (d\alpha^2 + d\beta^2 + d\gamma^2 + 2 \cos \beta d\alpha d\gamma),$$

$$\frac{\partial^2}{\partial u_\mu^2} = \frac{1}{u^3} \frac{\partial}{\partial u} \left(u^3 \frac{\partial}{\partial u} \right) - \frac{4}{u^2} \hat{j}^2,$$

where

$$\hat{j}^2 = -\frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left(\sin \beta \frac{\partial}{\partial \beta} \right) - \frac{1}{\sin^2 \beta} \left(\frac{\partial^2}{\partial \alpha^2} - 2 \cos \beta \frac{\partial^2}{\partial \alpha \partial \gamma} + \frac{\partial^2}{\partial \gamma^2} \right).$$

Thus, in terms of the coordinates (19) the isotropic oscillator is described by the equation

$$\frac{\partial^2 \Psi}{\partial u^2} + \frac{3}{u} \frac{\partial \Psi}{\partial u} - \frac{4}{u^2} \hat{j}^2 \Psi + \frac{2\mu}{\hbar^2} \left(E - \frac{\mu \omega^2 u^2}{2} \right) \Psi = 0.$$

The operators $\hat{J}^2, \hat{J}_3 = -i\partial/\partial\gamma, \hat{J}_{3'} = -i\partial/\partial\alpha$ are mutually commuting, and their eigenfunction is represented by the matrix of finite rotations

$$D_{ms}^j(\alpha, \beta, \gamma) = e^{im\alpha} d_{ms}^j(\beta) e^{is\gamma}.$$

The explicit form of the function $d_{ms}^j(\beta)$ is rather complicated, it can be found in manuals on Quantum Mechanics. It is important that the quantities j, m and s run the values $j = 0, 1/2, 1, \dots$ and $m, s = 0, \pm 1/2, \pm 1, \dots, \pm j$.

Now it is clear that the function Ψ should be of the form

$$\Psi = R(u) D_{ms}^j(\alpha, \beta, \gamma).$$

As the eigenvalues of the operator \hat{J}^2 are equal to $j(j+1)$, the radial function $R(u)$ satisfies the equation

$$\frac{d^2 R}{du^2} + \frac{3}{u} \frac{dR}{du} - \frac{4j(j+1)}{u^2} R + \frac{2\mu}{\hbar^2} \left(E - \frac{\mu \omega^2 u^2}{2} \right) R = 0.$$

This equation is solved as follows. First, use the dimensionless variable $v = au$ with $a = (\mu\omega/\hbar)^{1/2}$, and rewrite the last equation as

$$\frac{d^2 R}{dv^2} + \frac{3}{v} \frac{dR}{dv} - \frac{4j(j+1)}{v^2} R + \lambda R - v^2 R = 0,$$

where $\lambda = 2\mu E/\hbar^2 a^2 = 2E/\hbar\omega$. The next step is in the passage to the variable $\rho = v^2$ and the equation

$$\frac{d^2 R}{d\rho^2} + \frac{2}{\rho} \frac{dR}{d\rho} - \frac{j(j+1)}{\rho^2} R + \left(\frac{\lambda}{4\rho} - \frac{1}{4} \right) R = 0.$$

The analysis of this equation as $\rho \rightarrow 0$ and $\rho \rightarrow \infty$ verifies the appropriateness of the substitution

$$R(u) = \rho^j e^{-\rho/2} W(\rho)$$

leading to the equation for a confluent hypergeometric function

$$\rho \frac{d^2 W}{d\rho^2} + (2j+2-\rho) \frac{dW}{d\rho} - (j+1-\lambda/4)W = 0.$$

A further scenario is usual to any student who masters the course of Quantum Mechanics. The result is

$$W = F(j+1-\lambda/4, 2j+2; \rho),$$

$$j+1-\lambda/4 = -n, \quad n = 0, 1, 2, \dots$$

Concluding, we receive

$$E_N = \hbar\omega(N+2), \quad N = 2n+2j = 0, 1, 2, \dots \quad (20)$$

$$\Psi = Const(au)^{2j} e^{-a^2 u^2/2} F(-n, 2j+2, a^2 u^2) d_{ms}^j(\beta) e^{im\alpha} e^{is\gamma}. \quad (21)$$

For fixed j to the energy level E_N there correspond $(2j+1)^2$ states (degeneracy by m and s). As $j = \frac{N}{2}, \frac{N}{2}-1, \dots$, the total degeneracy for the N -th energy level is

$$g_N = \frac{1}{6} (N+1)(N+2)(N+3).$$

Observe now how the charge-dyon system could be obtained from the four-dimensional oscillator.

Using the variable $r = u^2$, we obtain

$$\begin{aligned} \frac{\partial^2}{\partial u_\mu^2} = & 4r \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left[\frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left(\sin \beta \frac{\partial}{\partial \beta} \right) + \frac{1}{\sin^2 \beta} \frac{\partial^2}{\partial \alpha^2} \right] \right. \\ & \left. + \frac{1}{r^2 \sin^2 \beta} \left[\frac{\partial^2}{\partial \gamma^2} - 2 \cos \beta \frac{\partial^2}{\partial \alpha \partial \gamma} \right] \right\}. \end{aligned}$$

Thus, the Schrödinger equation

$$\frac{d^2 \Psi}{du_\mu^2} + \frac{2\mu}{\hbar^2} \left(E - \frac{\mu\omega^2 u^2}{2} \right) \Psi = 0$$

gains (in terms of the coordinates (19)) the form

$$\begin{aligned} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r} \right) &+ \frac{1}{r^2} \left[\frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left(\sin \beta \frac{\partial \Psi}{\partial \beta} \right) + \frac{1}{\sin^2 \beta} \frac{\partial^2 \Psi}{\partial \alpha^2} \right] \\ &+ \frac{1}{r^2 \sin^2 \beta} \left[\frac{\partial^2 \Psi}{\partial \gamma^2} - 2 \cos \beta \frac{\partial^2 \Psi}{\partial \alpha \partial \gamma} \right] + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{e^2}{r} \right) \Psi = 0, \end{aligned}$$

with $e^2 = E/4$, $\varepsilon = -\mu\omega^2/8$.

Perform a substitution

$$\Psi(r, \alpha, \beta, \gamma) = \overline{\Psi}^{(s)}(r, \alpha, \beta) e^{is(\alpha+\gamma)}, \quad (22)$$

where s is any real parameter. It is easy to show that the function $\overline{\Psi}^{(s)}$ satisfies the equation

$$\begin{aligned} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \overline{\Psi}^{(s)}}{\partial r} \right) &+ \frac{1}{r^2} \left[\frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left(\sin \beta \frac{\partial \overline{\Psi}^{(s)}}{\partial \beta} \right) + \frac{1}{\sin^2 \beta} \frac{\partial^2 \overline{\Psi}^{(s)}}{\partial \alpha^2} \right] \\ &+ \frac{2is}{r^2(1 + \cos \beta)} \frac{\partial \overline{\Psi}^{(s)}}{\partial \alpha} - \frac{2s^2}{r^2(1 + \cos \beta)} \overline{\Psi}^{(s)} + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{e^2}{r} \right) \overline{\Psi}^{(s)} = 0. \end{aligned} \quad (23)$$

As in the previous sections, the expression for ε is easily found to be

$$\varepsilon = -\frac{\mu e^4}{2\hbar^2(n+j+1)^2}.$$

The wave function $\overline{\Psi}^{(s)}$ is obtained by comparing formulae (22) and (21), i.e.,

$$\overline{\Psi}^{(s)}(r, \alpha, \beta) = C \rho^j e^{-\rho/2} F(-n, 2j+2, \rho) d_{ms}^j(\beta) e^{i(m-s)\alpha},$$

where $\rho = a^2 u^2 = 2\mu e^2 r / \hbar^2 (n+j+1)$.

The first line of this equation is nothing but $\partial^2 \overline{\Psi}^{(s)} / \partial x_j^2$ where

$$x_1 + ix_2 = r \sin \beta e^{i\alpha}, \quad x_3 = r \cos \beta.$$

Then,

$$\frac{2is}{r^2(1 + \cos \beta)} \frac{\partial}{\partial \alpha} = \frac{is}{r^2(1 + \cos \beta)} \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha} \frac{is}{r^2(1 + \cos \beta)}.$$

Substituting here the formula

$$\frac{\partial}{\partial \alpha} = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1},$$

we obtain

$$\begin{aligned} \frac{2is}{r^2(1 + \cos \beta)} \frac{\partial}{\partial \alpha} &= \frac{isx_1}{r^2(1 + \cos \beta)} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_2} \frac{isx_1}{r^2(1 + \cos \beta)} \\ &- \frac{isx_2}{r^2(1 + \cos \beta)} \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_1} \frac{isx_2}{r^2(1 + \cos \beta)}. \end{aligned}$$

Also note that

$$\frac{s^2(x_1^2 + x_2^2)}{r^4(1 + \cos^2 \beta)} - \frac{2s^2}{r^2(1 + \cos \beta)} = -\frac{s}{r^2}.$$

Then, equation (23) can be rewritten as

$$\begin{aligned} \left(\frac{\partial}{\partial x_1} + \frac{isx_2}{r^2(1 + \cos \beta)} \right)^2 \bar{\Psi}^{(s)} + \left(\frac{\partial}{\partial x_2} - \frac{isx_1}{r^2(1 + \cos \beta)} \right)^2 \bar{\Psi}^{(s)} + \frac{\partial^2 \bar{\Psi}^{(s)}}{\partial x_3^2} \\ + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{e^2}{r} - \frac{\hbar^2 s^2}{2\mu r^2} \right) \bar{\Psi}^{(s)} = 0. \end{aligned}$$

This equation is identical to the Pauli equation

$$\left(\frac{\partial}{\partial x_j} - i \frac{e}{c} A_j \right)^2 \bar{\Psi}^{(s)} + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{e^2}{r} - \frac{\hbar^2 s^2}{2\mu r^2} \right) \bar{\Psi}^{(s)} = 0, \quad (24)$$

where the vector potential A_j is expressed as follows:

$$\vec{A} = \frac{g \sin \beta}{r(1 + \cos \beta)} (\sin \alpha, -\cos \alpha, 0) \quad (25)$$

with $g = \hbar cs/e$.

The vector potential (25) corresponds to the Dirac monopole with the magnetic charge g . So, equation (24) describes the motion for a charged particle e in the field of a dyon with charges $(-e, g)$. The presence of the charge $(-e)$ is indicated by the term e^2/r . The part $(-\hbar^2 s^2/2\mu r^2)$ presents a potential introduced by Goldhaber with the argument of conservation of the angular momentum in scattering of a charged particle from a magnetic monopole. As has been proved by Zwanziger, the addition of such a term makes a problem, corresponding to equation (24), superintegrable.

Thus, we are lucky to "synthesize" from the isotropic oscillator the bound charge-dyon system. It remains to clear up one important detail. As was shown by Dirac, the introduction of magnetic monopole in Quantum Mechanics leads to the quantization of an electric charge

$$e = \frac{\hbar c}{g} s, \quad s = 0, \pm 1/2, \pm 1, \pm 3/2, \dots$$

In our approach, the Dirac quantization condition is deduced from formula (22). The transformation $\gamma \rightarrow (\gamma + 4\pi)$ is identical, as it is seen from the coordinate definition (19). Requiring the single-valuedness for the wave function $\Psi(r, \alpha, \beta, \gamma)$, we come to the condition $s = 0, \pm 1/2, \pm 1, \dots$ which, together with the formula $g = \hbar cs/e$, leads to the quantization of an electric charge.

So, we have shown that the dyon-oscillator duality is valid for the four-dimensional oscillator.

Now focus on the duality transformation. So we have

$$x_1 + ix_2 = r \sin \beta e^{i\alpha} = 2r \sin(\beta/2) \cos(\beta/2) e^{i\alpha} = 2r \frac{u_1 + iu_2}{u} e^{-i(\alpha+\gamma)/2}$$

$$\begin{aligned}
& \cdot \frac{u_3 + iu_4}{u} e^{-i(\alpha-\gamma)/2} e^{i\alpha} = 2(u_1 + iu_2)(u_3 + iu_4), \\
x_3 &= r \cos \beta = r [\cos^2(\beta/2) - \sin^2(\beta/2)] = r \frac{u_1^2 + u_2^2}{u^2} - r \frac{u_3^2 + u_4^2}{u^2} \\
&= u_1^2 + u_2^2 - u_3^2 - u_4^2,
\end{aligned}$$

or, otherwise,

$$\begin{aligned}
x_1 &= 2(u_1 u_3 - u_2 u_4), \\
x_2 &= 2(u_1 u_4 + u_2 u_3), \\
x_3 &= u_1^2 + u_2^2 - u_3^2 - u_4^2.
\end{aligned}$$

This bilinear transformation satisfies Euler's condition $r = u^2$ and is called the Kustaanheimo–Stiefel transformation. It corresponds to the mapping $\mathbb{R}^4(\vec{u}) \rightarrow \mathbb{R}^3(\vec{x})$ that, along with the formula

$$\gamma = \frac{i}{2} \ln \left\{ \frac{u_1 + iu_2}{u_1 - iu_2} \frac{u_3 - iu_4}{u_3 + iu_4} \right\}$$

and the ansatz $\Psi \rightarrow \bar{\Psi}^{(s)}$, composes the duality transformation.

6 Magic Numbers

Let us answer the question why the dyon–oscillator duality is valid just for the oscillators with the configuration spaces of dimensions $D = 1, 2, 4, 8$. We have already mentioned that the duality transformation must satisfy the Euler's identity

$$(u_1^2 + u_2^2 + \dots + u_D^2)^2 = x_1^2 + x_2^2 + \dots + x_d^2, \quad (26)$$

where $d = 1$ for $D = 1$ and $d = D/2 + 1$ for $D > 1$. It was proved by Hurwitz that in the cases of x_i being a bilinear combination of u_i , the identity

$$(u_1^2 + u_2^2 + \dots + u_D^2)^2 = x_1^2 + x_2^2 + \dots + x_D^2 \quad (27)$$

is true for $D = 1, 2, 4, 8$. These magic numbers are directly related to the existence of the four fundamental algebraic structures: real numbers, complex numbers, quaternions and octonions. Putting in (27) $x_{d+1} = x_{d+2} = \dots = x_D = 0$, we come to (26).

7 Hurwitz Transformation

The question arises, of how to find a transformation converting $\mathbb{R}^8(u)$ into $\mathbb{R}^5(x)$, i.e. the transformation with the last of the magic numbers presented above. Begin to write down the transformation in the form

$$x = H(u; D) u.$$

Here D is the dimension of the space, H is the matrix $D \times D$ with the elements u_μ , and x, u are the D -dimensional columns composed from x_j, u_μ and, possibly, zeroes. So for the Levi–Civita and Kustaanheimo–Stiefel transformations, we have

$$\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \end{vmatrix},$$

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{vmatrix} = \begin{vmatrix} u_3 & -u_4 & u_1 & -u_2 \\ u_4 & u_3 & u_2 & u_1 \\ u_1 & u_2 & -u_3 & -u_4 \\ u_2 & -u_1 & -u_4 & u_3 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{vmatrix}.$$

The matrices $H(u; 2)$ and $H(u; 4)$ have the property

$$H(u; 2) H^T(u; 2) = u^2 E(2), \quad H(u; 4) H^T(u; 4) = u^2 E(4),$$

where "T" means the sign of transposition, $E(2)$ and $E(4)$ are the unit matrices. Due to these properties the Euler's identities are fulfilled. Now, one can easily deduce that the transformation $\mathbb{R}^8(\vec{u}) \rightarrow \mathbb{R}^5(\vec{x})$ must take the form

$$\begin{vmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} u_0 & u_1 & u_2 & u_3 & -u_4 & -u_5 & -u_6 & -u_7 \\ u_4 & u_5 & -u_6 & -u_7 & u_0 & u_1 & -u_2 & -u_3 \\ u_5 & -u_4 & u_7 & -u_6 & -u_1 & u_0 & -u_3 & u_2 \\ u_6 & u_7 & u_4 & u_5 & u_2 & u_3 & u_0 & u_1 \\ u_7 & -u_6 & -u_5 & u_4 & u_3 & -u_2 & -u_1 & u_0 \\ u_1 & -u_0 & u_3 & -u_2 & u_5 & -u_4 & u_7 & -u_6 \\ u_2 & -u_3 & -u_0 & u_1 & -u_6 & u_7 & u_4 & -u_5 \\ u_3 & u_2 & -u_1 & -u_0 & -u_7 & -u_6 & u_5 & u_4 \end{vmatrix} \begin{vmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{vmatrix}.$$

Whence it follows that

$$\begin{aligned} x_0 &= u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2, \\ x_1 &= 2(u_0 u_4 + u_1 u_5 - u_2 u_6 - u_3 u_7), \\ x_2 &= 2(u_0 u_5 - u_1 u_4 + u_2 u_7 - u_3 u_6), \\ x_3 &= 2(u_0 u_6 + u_1 u_7 + u_2 u_4 + u_3 u_5), \\ x_4 &= 2(u_0 u_7 - u_1 u_6 - u_2 u_5 + u_3 u_4). \end{aligned} \tag{28}$$

It is easy to prove that for the matrix $H(u; 8)$ there is a condition

$$H(u; 8) H^T(u; 8) = u^2 E(8)$$

that guarantees the validity of Euler's identity.

Adding to (28) the transformations

$$\begin{aligned}\alpha_T &= \frac{i}{2} \ln \frac{(u_0 + iu_1)(u_2 - iu_3)}{(u_0 - iu_1)(u_2 + iu_3)}, \\ \beta_T &= 2 \arctan \left(\frac{u_0^2 + u_1^2}{u_2^2 + u_3^2} \right)^{1/2}, \\ \gamma_T &= \frac{i}{2} \ln \frac{(u_0 - iu_1)(u_2 - iu_3)}{(u_0 + iu_1)(u_2 + iu_3)},\end{aligned}\tag{29}$$

we obtain a transformation converting \mathbb{R}^8 to the direct product $\mathbb{R}^5 \otimes \mathbf{S}^3$ of the space $\mathbb{R}^5(\vec{x})$ and a three-dimensional sphere $\mathbf{S}^3(\alpha_T, \beta_T, \gamma_T)$.

8 Yang–Coulomb Monopole

In the coordinates (28)-(29) the eight-dimensional isotropic oscillator is described by the equation

$$\frac{1}{2\mu} \left(-i\hbar \frac{\partial}{\partial x_j} - \hbar A_j^2 \hat{T}_a \right)^2 \Psi + \frac{\hbar^2}{2\mu r^2} \hat{T}^2 \Psi - \frac{e^2}{r} \Psi = \varepsilon \Psi \tag{30}$$

where ε and e^2 are defined as usual. The operators \hat{T}_a are the generators of the $SU(2)$ group. In the coordinates $(\alpha_T, \beta_T, \gamma_T)$ they are parametrized as follows:

$$\begin{aligned}\hat{T}^1 &= i \left(\cos \alpha_T \cos \beta_T \frac{\partial}{\partial \alpha_T} + \sin \alpha_T \frac{\partial}{\partial \beta_T} - \frac{\cos \alpha_T}{\sin \beta_T} \frac{\partial}{\partial \gamma_T} \right), \\ \hat{T}^2 &= i \left(\sin \alpha_T \cot \beta_T \frac{\partial}{\partial \alpha_T} - \cos \alpha_T \frac{\partial}{\partial \beta_T} - \frac{\sin \alpha_T}{\sin \beta_T} \frac{\partial}{\partial \gamma_T} \right), \\ \hat{T}^3 &= -i \frac{\partial}{\partial \alpha_T}.\end{aligned}$$

The five-dimensional vectors \vec{A}^a are given by the expressions

$$\begin{aligned}\vec{A}^1 &= \frac{1}{r(r+x_0)} (0, -x_4, -x_3, x_2, x_1), \\ \vec{A}^2 &= \frac{1}{r(r+x_0)} (0, x_3, -x_4, -x_1, x_2), \\ \vec{A}^3 &= \frac{1}{r(r+x_0)} (0, x_2, -x_1, x_4, -x_3).\end{aligned}$$

Each term of the triplet A_j^a coincides with the vector potential of a 5D Dirac monopole with a unit topological charge and with the line of singularity along the nonpositive x_0 semiaxis. The vectors A_j^a are orthogonal to each other

$$A_j^a A_j^b = \frac{1}{r^2} \frac{r - x_0}{r + x_0} \delta_{ab}$$

and to the vector $\vec{x} = (x_0, x_1, x_2, x_3, x_4)$ as well.

We see that equation (4) describes the charge–dyon system with $SU(2)$ monopoles which we call the Yang–Coulomb monopole (YCM). The YCM is defined as a five-dimensional system composed of the Yang monopole (A_j^a) of the topological charge +1 and the particle of the isospin (\hat{T}_a). Both the monopole and particle are also assumed to have electric charges of the opposite signs. Thus, the monopole–particle coupling is realized not only by the $SU(2)$ gauge field but also by the Coulomb interaction. At large distances the Coulomb structure becomes immaterial and YCM seems to be a pure Yang monopole. The YCM is a unique example of an integrable non-Abelian system. The $SO(6)$ group is a group of hidden symmetry of YCM which can be used for calculation of the energy spectrum of YCM by an algebraic method.

After quite complicated calculations, which are omitted here, we can reduce equation (30) to the form

$$\left(\Delta_5 - \frac{4}{r(r+x_0)} \hat{L} \hat{T} - \frac{2}{r(r+x_0)} \hat{T}^2 \right) \Psi + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{e^2}{r} \right) \Psi = 0, \quad (31)$$

where

$$\begin{aligned} \hat{L}_1 &= \frac{i}{2} [D_{41}(x) + D_{32}(x)], \\ \hat{L}_2 &= \frac{i}{2} [D_{13}(x) + D_{12}(x)], \\ \hat{L}_3 &= \frac{i}{2} [D_{12}(x) + D_{34}(x)] \end{aligned}$$

with

$$D_{ij} = -x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}.$$

We see that equation (31) contains the LT-coupling term demonstrating that we have no way to separate the wave function dependence on \mathbb{R}^5 and \mathbf{S}^3 .

In \mathbb{R}^5 we introduce the hyperspherical coordinates $r \in [0, \infty)$, $\theta \in [0, 2\pi]$, $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$ and $\gamma \in [0, 4\pi)$ according to the relations

$$\begin{aligned} x_0 &= r \cos \theta, \\ x_1 + ix_2 &= r \sin \theta \cos \frac{\beta}{2} e^{i \frac{\alpha+\gamma}{2}}, \\ x_3 + ix_4 &= r \sin \theta \sin \frac{\beta}{2} e^{i \frac{\alpha-\gamma}{2}}, \end{aligned}$$

and rewrite equation (31) as

$$\left(\Delta_{r\theta} - \frac{\hat{L}^2}{r^2 \sin^2(\theta/2)} - \frac{\hat{J}^2}{r^2 \cos^2(\theta/2)} \right) \Psi + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{e^2}{r} \right) \Psi = 0, \quad (32)$$

where $\hat{J}_a = \hat{L}_a + \hat{T}_a$ and

$$\Delta_{r\theta} = \frac{1}{r^4} \frac{\partial}{\partial r} \left(r^4 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^3 \theta} \frac{\partial}{\partial \theta} \left(\sin^3 \theta \frac{\partial}{\partial \theta} \right).$$

We introduce the separation ansatz

$$\Psi = \Phi(r, \theta) G(\alpha, \beta, \gamma; \alpha_T, \beta_T, \gamma_T),$$

where G are the eigenfunctions of \hat{L}^2 , \hat{T}^2 and \hat{J}^2 with the eigenvalues $L(L+1)$, $T(T+1)$ and $J(J+1)$.

Because of the LT-interaction, we seek the function G in the form

$$G = \sum_{M=m+t} (JM|L, m; T, t) D_{mm'}^L(\alpha, \beta, \gamma) D_{tt'}^T(\alpha_T, \beta_T, \gamma_T),$$

where $(JM|L, m; T, t)$ are the Clebsch–Gordan coefficients.

Let us take the function $\Phi(r, \theta)$ in the form

$$\Phi(r, \theta) = R(r)Z(\theta).$$

Equation (32) is then separated into

$$\frac{1}{\sin^3 \theta} \frac{d}{d\theta} \left(\sin^3 \theta \frac{dZ}{d\theta} \right) - \frac{2L(L+1)}{1 - \cos \theta} Z - \frac{2J(J+1)}{1 + \cos \theta} Z + \lambda(\lambda+3)Z = 0 \quad (33)$$

and a purely radial equation

$$\frac{1}{r^4} \frac{d}{dr} \left(r^4 \frac{dR}{dr} \right) - \frac{\lambda(\lambda+3)}{r^2} + \frac{2\mu}{\hbar^2} \left(\varepsilon + \frac{e^2}{r} \right) R = 0 \quad (34)$$

with the separation constant $\lambda(\lambda+3)$ being equal to the nonnegative eigenvalues of the global angular momentum.

In equation (33) it is convenient to change the variable as $y = (1 - \cos \theta)/2$ and set

$$Z(y) = y^L (1 - y)^J W(y).$$

Substituting this into equation (33) we obtain the hypergeometric equation

$$y(1-y) \frac{d^2 W}{dy^2} + [c - (a+b+1)y] \frac{dW}{dy} - abW = 0,$$

where $a = -\lambda + L + J$, $b = \lambda + L + J + 3$, $c = 2L + 2$.

Thus, we find that

$$Z(\theta) = (1 - \cos \theta)^L (1 + \cos \theta)^L F \left(-\lambda + J + L, \lambda + J + L + 3, 2L + 2; \frac{1 - \cos \theta}{2} \right).$$

The solution behaves well at $\theta = \pi$ if the series F terminates, that is

$$-\lambda + J + L = -n_\theta,$$

with $n_\theta = 0, 1, 2, \dots$

Let us now consider the radial equation and introduce the function

$$f(r) = e^{-kr} r^{-\lambda} R(r).$$

It can easily be verified that the equation for $f(r)$ has the form of the confluent hypergeometric equation

$$z \frac{d^2 f}{dz^2} + (c - z) \frac{df}{dz} - af = 0,$$

where $z = 2kr$, $k = \sqrt{-2\mu\varepsilon/\hbar^2}$, $c = 2\lambda + 4$, $a = \lambda + 2 - 1/kr_0$, $r_0 = \hbar^2/me^2$. For the bound state solutions ($\varepsilon < 0$), we have

$$\lambda + 2 - 1/kr_0 = -n_r, \quad n_r = 0, 1, 2, \dots;$$

therefore,

$$\varepsilon_N^T = -\frac{me^4}{2\hbar^2(N/2 + 2)^2},$$

where $N = 2(n_r + \lambda) = 2(n_r + n_\theta + J + L)$.

For fixed T , the energy levels ε_N^T do not depend on L , J and λ , i.e., they are degenerate. The total degeneracy is

$$g_N^T = (2T + 1) \sum_{\lambda} \sum_L (2L + 1) \sum_J (2J + 1).$$

After some tedious calculation we finally obtain

$$g_N^T = \frac{1}{12}(2T+1)^2 \left(\frac{N}{2} - T + 1\right) \left(\frac{N}{2} - T + 2\right) \left\{ \left(\frac{N}{2} - T + 2\right) \left(\frac{N}{2} - T + 3\right) + 2T(N + 5) \right\}.$$

For $T = 0$ and $N = 2n$ (even) the right-hand side of the last formula is equal to $(n + 1)(n + 2)^2(n + 3)/12$ – that is, to the degeneracy of pure Coulomb levels. Further, we have $T = 0, 1, \dots, N/2$ for even N and $T = 1/2, 3/2, \dots, N/2$ for odd N . Therefore,

$$g_N = \sum_{T=0, \frac{1}{2}}^{N/2} g_N^T = \frac{(N + 7)!}{7!N!}$$

i.e., we obtain the degeneracy of the energy levels for the 8D isotropic quantum oscillator.

Formulae (28) and (29) represent the duality transformation mapping the 8D quantum oscillator into charge–dyon system with the $SU(2)$ monopole.

9 Oscillator-like Systems

We have considered above the dyon–oscillator duality. This type of duality is valid not only for the $1D, 2D, 4D$ and $8D$ oscillators, but also for oscillator-like systems with the potentials

$$V(u^2) = C_0 + C_2 u^2 + W(u^2),$$

where $W(u^2)$ has the form

$$W(u^2) = \sum_{n=2}^{\infty} C_{2n} u^{2n}$$

For such modified potentials the ansatz (4) can be rewritten as

$$\varepsilon = -\frac{C_2}{4}, \quad e^2 = \frac{E - C_0}{4}$$

Thus, the value of the function $V(u^2)$ at $u^2 = 0$ contributes to the Coulomb coupling constant e^2 . It is also easy to verify that the left-hand side of equation (34) develops the additional term $(-W(r)/4r)$.

10 Exercises

- Find the energy levels and normalized wave functions for states of the particle placed in the field $V(x) = -\alpha/|x| - \hbar^2 \nu(\nu - 1)/2\mu x^2$, where $x \in (-\infty, \infty)$ and $\nu \neq 0, 1/2$ (see Ref. [50]).
- Prove that the 3D oscillator with coordinates confined by the 2D half-up cone for an angle of $\pi/6$ is dual to the 2D charge–dyon system obeying fractional statistics. Find the duality transformation (see Ref. [48]).
- Calculate the length-element dl^2 , metric tensor $g_{\mu\nu}$ and the Laplace operator $\partial^2/\partial u_\mu^2$ in the coordinates (19).
- Compute the integrals of motion for the 3-dimensional charge–dyon system, transforming for $g = 0$ into the operator of orbital momentum and Runge–Lenz operator (see Ref. [45]).
- Prove that the Goldhaber correction in the Hamiltonian of the 3-dimensional charge–dyon system is identical to the interaction $\vec{\mu}\vec{B}$ of the magnetic momentum $\vec{\mu}$ of a particle with the magnetic field \vec{B} (see Ref. [45]).
- Solve the Schrödinger equation for the 3-dimensional charge–dyon system in the parabolic coordinates $x_1 = \sqrt{\xi\eta} \cos \varphi$, $x_2 = \sqrt{\xi\eta} \sin \varphi$, $x_3 = (\xi - \eta)/2$ (see Ref. [49]).
- Compute the expansion coefficients of the parabolic basis of the 3-dimensional charge–dyon system in terms of its spherical basis (see Ref. [49]).

- Prove that the transformation (28) converts the Schrödinger equation for the $8D$ oscillator into equation (30).
- Show that equation (30) can be transformed into (31).
- Calculate the length-element dl^2 , metric tensor g_{ij} and Laplace operator $\partial^2/\partial x_j^2$ in the coordinates $(r, \theta, \alpha, \beta, \gamma)$.

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